DHANAMANJURI UNIVERSITY
DECEMBER-2025

Name of Programme : M.A./M.Sc. Mathematics
Semester : 1st
Paper Code : MAT-501
Paper Title : Advanced Abstract Algebra-I
Full Marks : 80
Pass Marks : 32 **Duration: 3 hours**

The figures in the margin indicate full marks for the questions.

Answers all the questions:

UNIT-I

A. Answer any three from the following questions: **$10 \times 3 = 30$**

1. a) Prove that any finite group G with at least two elements has a maximal normal subgroup.

   b) Let H and K be two distinct normal subgroups of a group G, then show that $G = HK$ and $H \cap K$ is a maximal normal subgroup of H as well as K. 4+6=10

2. State and prove Jordan-Holder theorem. 1+9=10

3. Define solvable group and give an example. Prove that a group G is solvable iff $G^{(n)} = \{e\}$ for some positive integer n, where $G^{(n)}$ denotes the n^{th} derived group of G. Also, show that a subgroup of a solvable group is solvable. 2+6+2=10

4. a) Show that a finite group is solvable if and only if its composition factors are cyclic groups of prime orders.

   b) Prove that every nilpotent group is solvable but the converse is not true. 5+5=10

5. a) Prove that any subgroup of a nilpotent group is nilpotent.

   b) If H and K are nilpotent groups then show that $H \times K$ is also nilpotent. 5+5=10

UNIT-II

B. Answer any three from the following questions: $10 \times 3 = 30$

1. a) Define an algebraic element.

   b) Differentiate monic polynomial from minimal polynomial.

   c) If an element $a \in K$ is algebraic over F, then show that there exists a unique monic polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Further, if $f(x) \in F[x]$ be a polynomial such that $f(a) = 0$, then show that $p(x) | f(x)$. 1+1+5+3=10

2. Let K be an extension field of a field F and let $a \in K$ be an algebraic of degree n. Prove that $F(a) = \{\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \dots + \alpha_{n-1} a^{n-1} \mid \alpha_i \in F, \forall\, i = 0, 1, 2, \dots, n-1\}$ is a subfield of K containing F and a.

3. a) Define an algebraic extension field of a field F.

   b) Show that every algebraic extension field of a field K is a finite extension field but the converse does not hold true. 2+8=10

4. Prove that a set S of all those elements of K which are algebraic over a field F forms a subfield of K containing F such that no element of K, not in S is algebraic over S.

5. If L is an algebraic extension of a field K and K is an algebraic extension of a field F then show that L is an algebraic extension of the field F.

UNIT-III

C. Answer any two from the following questions: $10 \times 2 = 20$

1. a) Let E be a Galois extension of a field F. Let K be any subfiled of E containing F. Then, show that the mapping $K \to G(E/K)$ set up a one-one correspondence from the set of subfields of E containing F to the subgroups of $G(E/F)$ such that

   i) $K = E_{G(E/K)}$

   ii) For any subgroup H of $G(E/F)$, $H = G(E/E_H)$ 4+4=8

b) Let F be a field of characteristic $\neq 2$. Let $x^2 - a \in F[x]$ be an irreducible polynomial over F, then prove that its Galois group is of order 2. 2

2\. Prove that every polynomial $f(x) \in C[x]$ factors into linear factors in $C[x]$.

a) Prove that $x^n - 1 = \prod_{d|n} \Phi_d(x)$, where $\Phi_d(x)$ denotes the d^{th} cyclotomic polynomial over the field K.

b) Show that a polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over $\mathbb{Q}$. 5+5=10

*****